Observations Regarding the Collatz Conjecture

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2022-07-22

Document version: 2022-08-08

1 Abstract

This is a loose collection of definitions and formulas useful when researching the Collatz conjecture. There is nothing new in this document. The results herein have been discovered by others earlier and they are so widely known that they can be considered common knowledge. I use this document as my personal point of reference.

2 Collatz Functions and Sequences

Definition 2.1 (Original Collatz Function)**.** The original Collatz function (or original Collatz map, or simply "the Collatz map") is

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} .$$

Definition 2.2 (Original Collatz Sequence / OCS)**.** A Collatz sequence, or original Collatz sequence (OCS), is the sequence $(C^i(n_0))_{i \in \mathbb{N}_0}$, where $n_0 \in \mathbb{N}$ is the starting value and C^i is the i-times iterated function C.

Definition 2.3 (Shortcut Collatz Function)**.** The shortcut Collatz function is

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} .$$

Definition 2.4 (Shortcut Collatz Sequence / SCS)**.** A shortcut Collatz sequence (SCS) is the sequence $(T^i(n_0))_{i\in\mathbb{N}_0}$, where $n_0 \in \mathbb{N}$ is the starting value and T^i is the i-times iterated function T.

Remark 2.5. An SCS is similar to the OCS which starts at the same starting number n_0. The difference of the SCS is that it leaves out the even number that comes directly after every odd number in the OCS. This is because for odd n, the term $3n+1$ always evaluates to an even number, which is always divided by 2 in the next operation. The shortcut Collatz function integrates this division by 2 directly into the odd rule.

Remark 2.6. Note that $T(n)$ does a division by 2 exactly once in every application, so in the iterated function $T^i(n)$ there happen i many divisions by 2.

3 Cycles

Definition 3.1 (Cycle Member in an SCS)**.** The number $n \in \mathbb{N}$ is a cycle member in an SCS if, and only if, the sequence starting at n reaches this number again, i.e. iff $\exists m \in \mathbb{N} : T^m(n) = n$.

Remark 3.2. Note that $m = 0$ is excluded by $m \in \mathbb{N}$.

Remark 3.3. For example $T(1) = 2$ and $T(2) = 1$, so $T^2(1) = T(T(1)) = 1$ and $T^2(2) = 2$. Thus $\{1, 2\}$ are cycle members in an SCS. This is the only known positive cycle in all SCSes and it is easy to find, thus it is called the trivial Collatz cycle, or simply the trivial cycle.

4 Parity Vector

Definition 4.1 (Parity Vector)**.** Let $n \in \mathbb{N}$. Each element v_i of the parity vector $v(n)$ is defined as

$$v_i(n) := \begin{cases} 1 & \text{if } T^i(n) \text{ is odd} \\ 0 & \text{if } T^i(n) \text{ is even} \end{cases}, \text{ where } i \in \mathbb{N}_0.$$

5 Iteration Speedups for Special $n \in \mathbb{N}$

Certain forms of $n \in \mathbb{N}$ allow to skip ahead many iterations at once. Some are shown below.

Lemma 5.1 (Single Iteration Binary 1's Speedup)**.** *Let $n \in \mathbb{N}$ with $n = h \cdot 2^s + (2^s - 1)$, where $h \in \mathbb{N}_0$, $s \in \mathbb{N}$. Then $T(n) = (3h + 2)2^{s-1} + (2^{s-1} - 1)$.*

Proof. Let $n = h \cdot 2^s + (2^s - 1)$, where $h \in \mathbb{N}_0$, $s \in \mathbb{N}$.

Then $n = h \cdot 2^s + 2^s - 1 = 2^s(h+1) - 1$. Note that n is odd, because 2^s is even and so is $2^s(h+1)$, so subtracting 1 from this makes the result odd.

$$\begin{aligned}
T(n) &= \frac{3n+1}{2} \\
&= \frac{3(2^s(h+1)-1)+1}{2} \\
&= \frac{3 \cdot 2^s(h+1) - 3 + 1}{2} \\
&= \frac{3 \cdot 2^s(h+1) - 2}{2} \\
&= \frac{2 \cdot (3 \cdot 2^{s-1}(h+1) - 1)}{2} \\
&= 2^{s-1}3(h+1) - 1 \\
&= 2^{s-1}(3h+3) - 1 \\
&= 2^{s-1}(3h+2+1) - 1 \\
&= 2^{s-1}(3h+2) + 2^{s-1} - 1 \\
&= (3h+2)2^{s-1} + (2^{s-1} - 1)
\end{aligned}$$

□

Proposition 5.2 (Multiple Iterations Binary 1's Speedup)**.** Let $n \in \mathbb{N}$ with $n = h \cdot 2^s + (2^s - 1)$, where $h, s \in \mathbb{N}_0$. Then $T^s(n) = h \cdot 3^s + (3^s - 1)$.

Proof. Let $n = h \cdot 2^s + (2^s - 1)$, where $h, s \in \mathbb{N}_0$.

Case by case analysis of the cases $s = 0$ and $s > 0$:

Case $s = 0$:

$$\begin{aligned}
T^0(n) &= n \\
&= h \cdot 2^s + (2^s - 1) \\
&= h \cdot 2^0 + (2^0 - 1) \\
&= h \cdot 1 + (1 - 1) \\
&= h \cdot 3^0 + (3^0 - 1) \\
&= h \cdot 3^s + (3^s - 1)
\end{aligned}$$

Case $s > 0$:

$s > 0 \implies s \in \mathbb{N}$, so lemma 5.1 can be applied:

$$\begin{aligned}T^s(n) &= T^{s-1}(T(n)) \\ &= T^{s-1}(T(h \cdot 2^s + 2^s - 1)) \\ &= T^{s-1}((3h+2)2^{s-1} + (2^{s-1} - 1))\end{aligned}$$

Now with $h' = 3h + 2$, lemma 5.1 is applied another time, which results in

$$\begin{aligned}T^s(n) &= T^{s-1}((3h+2)2^{s-1} + (2^{s-1} - 1)) \\ &= T^{s-2}((3(3h+2)+2)2^{s-2} + (2^{s-2} - 1)) \\ &= T^{s-2}((3^2h + 3^1 \cdot 2 + 3^0 \cdot 2)2^{s-2} + (2^{s-2} - 1))\end{aligned}$$

When applying lemma 5.1 multiple times, a regularity shows as follows.

$$\begin{aligned}T^s(n) &= T^{s-2}((3^2h + 3^1 \cdot 2 + 3^0 \cdot 2)2^{s-2} + (2^{s-2} - 1)) \\ &= T^{s-3}(3((3^2h + 3^1 \cdot 2 + 3^0 \cdot 2) + 2)2^{s-3} + (2^{s-3} - 1)) \\ &= T^{s-3}((3^3h + 3^2 \cdot 2 + 3^1 \cdot 2 + 3^0 \cdot 2)2^{s-3} + (2^{s-3} - 1)) \\ &\ldots \\ &= T^{s-s}((3^sh + 3^{s-1} \cdot 2 + 3^{s-2} \cdot 2 + \ldots + 3^0 \cdot 2)2^{s-s} + (2^{s-s} - 1)) \\ &= T^0((3^sh + 2 \cdot (3^{s-1} + 3^{s-2} + \ldots + 3^0))2^0 + (2^0 - 1)) \\ &= h \cdot 3^s + 2 \cdot (3^{s-1} + 3^{s-2} + \ldots + 3^0) \\ &= h \cdot 3^s + 2\sum_{i=0}^{s-1} 3^i \\ &= h \cdot 3^s + 2\frac{3^{(s-1)+1} - 1}{3 - 1} = 3^sh + 2\frac{3^s - 1}{2} \\ &= h \cdot 3^s + 3^s - 1\end{aligned}$$

□

Corollary 5.3. *Let $n \in \mathbb{N}$ with $n = 2^s - 1$, where $s \in \mathbb{N}_0$. Then $T^s(n) = 3^s - 1$.*

Proof. Let $n = 2^s - 1$, where $s \in \mathbb{N}_0$. Then with $h = 0$ we can rewrite this as $n = 2^s - 1 = h \cdot 2^s + (2^s - 1)$ and with proposition 5.2 it follows that $T^s(n) = h \cdot 3^s + (3^s - 1) = 3^s - 1$. □

Proposition 5.4. Let $n \in \mathbb{N}$ with $n = (h \cdot 2^s + (2^s - 1)) \cdot 2^z$, where $h, s, z \in \mathbb{N}_0$. Then $T^{s+z}(n) = h \cdot 3^s + (3^s - 1)$.

Proof. Let $n = (h \cdot 2^s + (2^s - 1)) \cdot 2^z$, where $h, s, z \in \mathbb{N}_0$.

Because n can be divided z times by 2, it follows that

$$T^z(n) = n \cdot \frac{1}{2^z} = (h \cdot 2^s + (2^s - 1)) \cdot 2^z \cdot \frac{1}{2^z} = h \cdot 2^s + (2^s - 1) \ .$$

From this follows with proposition 5.2 that

$$\begin{aligned} T^{s+z}(n) &= T^s(T^z(n)) \\ &= T^s(h \cdot 2^s + (2^s - 1)) \\ &= h \cdot 3^s + 3^s - 1 \ . \end{aligned}$$

□

Corollary 5.5. *Let $n \in \mathbb{N}$ with $n = (2^s - 1) \cdot 2^z$, where $s, z \in \mathbb{N}_0$. Then $T^{s+z}(n) = 3^s - 1$.*

Proof. Let $n = (2^s - 1) \cdot 2^z$, where $s \in \mathbb{N}_0$. Then with $h = 0$ we express n as $n = (2^s - 1) \cdot 2^z = (h \cdot 2^s + (2^s - 1)) \cdot 2^z$ and with proposition 5.4 it follows that $T^{s+z}(n) = h \cdot 3^s + (3^s - 1) = 3^s - 1$. □

6 Decomposing T

Remark 6.1 (Functions of Starting Value n and Index i)**.** In the following, the sequences p_i, q_i, a_i, b_i, α_i, and β_i will be defined. Actually, these are functions of not only their indices, but also of the starting value n. So technically the consistent notations for these are $p(n, i)$, $q(n, i)$, and so forth.
However, this notation would clutter formulas and make them difficult to read. To mitigate this, n is assumed to be known from the context and fixed. Therefor, n is not mentioned as an argument to these functions. Moreover, i (or j, k, ...) is written as an index rather than as a function argument, so parentheses can be widely omitted to help readability. Hence, it is written p_i instead of $p(n, i)$, q_i instead of $q(n, i)$, and so on.

Definition 6.2 (Number of Odd Steps)**.** Let $n \in \mathbb{N}$. Then p_i is the count of odd elements in the tuple $(T^j(n))_{j \in \mathbb{N}_0, j<i}$.

Definition 6.3 (Number of Even Steps)**.** Let $n \in \mathbb{N}$. Then q_i is the count of even elements in the tuple $(T^j(n))_{j \in \mathbb{N}_0, j<i}$.

Remark 6.4. Note that $p_i + q_i = i$ is the number of iterations in $T^i(n)$, because $p_i + q_i$ is the added counts of odd elements and even elements, so the count of all elements, in $(T^j(n))_{j \in \mathbb{N}_0, j<i}$. These are i many.

Definition 6.5 (Factors and Biases, Single Step)**.** Let $n \in \mathbb{N}$. Then n is the starting value of the SCS $(T^i(n))_{i \in \mathbb{N}_0}$.

Define

$$a_k := \begin{cases} \frac{3}{2} & \text{if } T^k(n) \text{ is odd} \\ \frac{1}{2} & \text{if } T^k(n) \text{ is even} \end{cases} \text{, with } k \in \mathbb{N}_0,$$

and

$$b_j := \begin{cases} \frac{1}{2} & \text{if } T^j(n) \text{ is odd} \\ 0 & \text{if } T^j(n) \text{ is even} \end{cases} \text{, with } j \in \mathbb{N}_0.$$

Definition 6.6 (Factors and Biases, Multiple Steps)**.** Let $n \in \mathbb{N}$, $i \in \mathbb{N}_0$. Then define

$$\alpha_i := \prod_{k=0}^{i-1} a_k \tag{6.1}$$

and

$$\beta_i := \sum_{j=0}^{i-1} (b_j \prod_{k=j+1}^{i-1} a_k) \ . \tag{6.2}$$

Remark 6.7. Note that a_i and b_i are defined based on the i'th iterated function T. Unlike these, α_i and β_i are based on the iterations only up to T^{i-1}, i.e. they are "ready" one step before T^i. Therefor α_i and β_i can be used to express T^i, as shown by the next proposition.

Proposition 6.8 (Decomposition)**.** Let $n \in \mathbb{N}$, $i \in \mathbb{N}_0$. Then $T^i(n) = \beta_i + \alpha_i n$.

Proof. Let $n \in \mathbb{N}$, $i \in \mathbb{N}_0$.

We'll do a case by case analysis of the cases $i = 0$ and $i > 0$ and see that both conclude to the equation in the assertion.

Case $i = 0$:
$\beta_i = \beta_0 = 0$, because it is an empty sum. $\alpha_i = \alpha_0 = 1$, because it is an empty product. Then it follows that $T^i(n) = T^0(n) = n = 0 + 1 \cdot n = \beta_i + \alpha_i n$.

Case $i > 0$:

$$
\begin{aligned}
T^i(n) &= b_{i\text{-}1} + a_{i\text{-}1}T^{i\text{-}1}(n) \\
&= b_{i\text{-}1} + a_{i\text{-}1}(b_{i\text{-}2} + a_{i\text{-}2}T^{i\text{-}2}(n)) \\
&= b_{i\text{-}1} + a_{i\text{-}1}b_{i\text{-}2} + a_{i\text{-}1}a_{i\text{-}2}T^{i\text{-}2}(n) \\
&= b_{i\text{-}1} + a_{i\text{-}1}b_{i\text{-}2} + a_{i\text{-}1}a_{i\text{-}2}(b_{i\text{-}3} + a_{i\text{-}3}T^{i\text{-}3}(n)) \\
&= b_{i\text{-}1} + a_{i\text{-}1}b_{i\text{-}2} + a_{i\text{-}1}a_{i\text{-}2}b_{i\text{-}3} + a_{i\text{-}1}a_{i\text{-}2}a_{i\text{-}3}T^{i\text{-}3}(n) \\
&= ... \\
&= b_{i\text{-}1} + a_{i\text{-}1}b_{i\text{-}2} + a_{i\text{-}1}a_{i\text{-}2}b_{i\text{-}3} + ... + a_{i\text{-}1}a_{i\text{-}2}...a_1b_0 + a_{i\text{-}1}a_{i\text{-}2}...a_0T^{i-i}(n) \\
&= b_{i\text{-}1} + a_{i\text{-}1}b_{i\text{-}2} + a_{i\text{-}1}a_{i\text{-}2}b_{i\text{-}3} + ... + a_{i\text{-}1}a_{i\text{-}2}...a_1b_0 + a_{i\text{-}1}a_{i\text{-}2}...a_0n \\
&= \underbrace{\sum_{j=0}^{i-1}(b_j \prod_{k=j+1}^{i-1} a_k)}_{\beta_i} + \underbrace{(\prod_{k=0}^{i-1} a_k)}_{\alpha_i}n \\
&= \beta_i + \alpha_i n
\end{aligned}
$$

□

Lemma 6.9. *If $n \in \mathbb{N}$, $i \in \mathbb{N}_0$, then $p_i, q_i, \beta_i \geq 0$ and $\alpha_i > 0$.*

Proof.
$p_i, q_i \geq 0$, because they are counts of elements, so they are ≥ 0 by definition.

$\beta_i \geq 0$, because β_i is a sum of terms of which each is ≥ 0, because $b_j \geq 0$ and the product of a_k's is > 0, because each $a_k > 0$.

$\alpha_i > 0$, because by definition it's a product of $a_k > 0$. □

Lemma 6.10. *Let $n \in \mathbb{N}$. Then $\alpha_i = \frac{3^{p_i}}{2^i}$.*

Proof. $\alpha_i = \prod_{k=0}^{i-1} a_k$, where $a_k := \begin{cases} \frac{3}{2} & \text{if } T^k(n) \text{ is odd} \\ \frac{1}{2} & \text{if } T^k(n) \text{ is even} \end{cases}$.

That means that the factors a_k are p_i many times $\frac{3}{2}$ and q_i many times $\frac{1}{2}$, so

$$\alpha_i = \prod_{k=0}^{i-1} a_k = (\frac{3}{2})^{p_i} \cdot (\frac{1}{2})^{q_i} = \frac{3^{p_i}}{2^{p_i}} \cdot \frac{1}{2^{q_i}} = \frac{3^{p_i}}{2^{p_i+q_i}} = \frac{3^{p_i}}{2^i}$$

. □

Proposition 6.11. Let $n, i \in \mathbb{N}_0, i > 0$. Then $\alpha_i \neq 1$.

Proof. By lemma 6.10 $\alpha_i = \frac{3^{p_i}}{2^i}$. By the fundamental theorem of arithmetic, the representation of a number by its prime factors is unique. The numbers 3^{p_i} and 2^i have different prime factors, so they are not equal. And then $3^{p_i} \neq 2^i \implies \frac{3^{p_i}}{2^i} \neq 1$ and with lemma 6.10 it follows that $\alpha_i = \frac{3^{p_i}}{2^i} \neq 1$. □

Corollary 6.12. *If $n \in \mathbb{N}$, $i \in \mathbb{N}_0$, then $\alpha_i > 1 \implies T^i(n) > n$.*

Proof. Let $\alpha_i > 1$. By proposition 6.8 $T^i(n) = \beta_i + \alpha_i n \geq \alpha_i n > n$. □

Corollary 6.13. *If $n \in \mathbb{N}$, $i \in \mathbb{N}_0, i > 0$, then $T^i(n) \leq n \implies \alpha_i < 1$.*

Proof. Let $n \in \mathbb{N}$, $i \in \mathbb{N}_0, i > 0$. Then $T^i(n) \leq n \implies \alpha_i \leq 1$ is already proved, because it's the contrapositive of corollary 6.12. With $\alpha_i \neq 1$ from proposition 6.11 it follows that $T^i(n) \leq n \implies \alpha_i < 1$. □

Remark 6.14. $\alpha_i < 1$ does not generally imply $T^i(n) < n$. First of all, this cannot be generally true, because we know a counter example: The trivial cycle with $n = 1$, for which we calculate that $\alpha_2 = \frac{3}{4} < 1$, but $T^2(1) = 1 = n$, so $T^2(n) \not< n$.
The mentioned implication does hold for a lot of cases, e.g. it holds for all $4615 \leq n \leq 10^{10}$ and $i \in \mathbb{N}_0$, but there are also many counter examples for $n \leq 4614$.
The smallest counter example that is not a member of the trivial cycle is $n = 7, i = 8$ with $T^8(7) = 8$ with $\alpha_8 = \frac{3^5}{2^8} \approx 0.9492$. The largest one that I know is $n = 4614, i = 73$ with $T^{73}(4614) = 4616$ with $\alpha_{73} = \frac{3^{46}}{2^{73}} \approx 0.9384$.

The table below gives an overview of the possibilities permitted by the above corollaries. The questionmark (?) is meant to reflect that there might be more examples for this case that I don't know about.

Table 1: Cases Overview for Corollaries 6.12 and 6.13, $i > 0$

$\wedge$	$\alpha_i < 1$	$\alpha_i > 1$
$T^i(n) < n$	usual	impossible
$T^i(n) = n$	$n \in \{1, 2, ?\}$	impossible
$T^i(n) > n$	known counter examples, e.g. for $n \in \{7, 9, 18, 19, 25, 73, (...,)\ 4613, 4614, ?\}$	usual

7 Notation

$\mathbb{N}$	natural numbers, not including 0
$\mathbb{N}_0$	natural numbers, including 0
$\wedge$	logical and
$:$	read “so that”, “for which it holds that”, or simply “with”, e.g. in $\exists m \in \mathbb{N} : T^m(n) = n$